

# A Robust Solution to Erdős Problem #460

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## Abstract

Fix  $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ . Define a sequence  $(a_k)_{k \geq 0}$  by  $a_0 = n$ ,  $a_1 = 1$ , and for each  $k \geq 2$  let  $a_k$  be the **least** integer  $m > a_{k-1}$  such that

$$\gcd(n - m, n - a_i) = 1 \quad \text{for all } 1 \leq i < k.$$

Defining  $b_i := n - a_i$ , it follows from the construction that each new difference  $b_k$  is coprime to all previous  $b_i$ , hence the family  $(b_i)_{i \geq 1}$  is pairwise coprime. Define the associated reciprocal series

$$S(n) := \sum_{i \geq 1} \frac{1}{a_i}.$$

Erdős posed Problem #460 in the context of greedy coprimality constructions in combinatorial number theory (see, e.g., [Erd77, EG80]), and a current statement is recorded in the Erdős Problems database [Blo]. The problem asks whether  $S(n) \rightarrow \infty$  as  $n \rightarrow \infty$ , and also asks for the behavior of the subseries obtained by restricting to indices  $i$  for which  $n - a_i$  has a prime divisor  $p \leq a_i$  (and of its complementary subseries).

We give a complete answer:

- For  $n = 1, 2$ , the greedy process terminates and  $S(n) < \infty$ .
- For every  $n \geq 3$ , the greedy recursion is well-defined and infinite, and  $S(n) = +\infty$  (diverges). Consequently, in the extended-real sense,  $S(n) \rightarrow \infty$  as  $n \rightarrow \infty$ .
- For every  $n \geq 3$ , the “good-index” restricted sum

$$S_{\leq}(n) := \sum_{\substack{i \geq 1: \\ \exists p \text{ prime, } p \leq a_i, p | (n - a_i)}} \frac{1}{a_i}$$

also diverges to  $+\infty$ .

- For every  $n \in \mathbb{N}$ , the complementary “bad-index” subseries

$$S_{>}(n) := \sum_{\substack{i \geq 1: \\ \forall p \text{ prime, } p \leq a_i, p \nmid (n - a_i)}} \frac{1}{a_i}$$

is finite (hence convergent).

The main engine is an *embedded prime subsequence*: for each  $n \geq 3$  and each prime  $q > n - 1$ , the term  $a = n + q$  must occur in the greedy sequence, yielding a lower bound for  $S(n)$  (and for  $S_{\leq}(n)$ ) by a shifted tail of the divergent reciprocal-primes series. For the clean comparison inequality  $1/(n + q) > 1/(2q)$  we sum over primes  $q > n$ , avoiding the single boundary possibility  $q = n$  when  $n$  is prime.

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## 1 Executive summary

For  $n \geq 3$ , the greedy coprimality condition forces the difference values  $b_i := n - a_i$  to be pairwise coprime and nonzero. This makes it impossible to “avoid”  $b = -q$  once  $q$  is a sufficiently large prime: any earlier  $b_i$  is too small in absolute value (and nonzero) to be divisible by  $q$ . Therefore  $a = n + q$  must occur for every prime  $q > n - 1$ . The sum  $S(n)$  then dominates a shifted tail of  $\sum_{q \text{ prime}} 1/q$ , which diverges. A technical rigor point is that the clean inequality  $1/(n + q) \geq (1/2)(1/q)$  is used only for primes  $q > n$ .

## 2 Foundational setup

### 2.1 Universe and primitives

We work in  $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $\mathbb{Z}$ . Primitive relations:

- divisibility  $d \mid x$  for  $d, x \in \mathbb{Z}$ ;

- $\gcd(x, y)$  for  $x, y \in \mathbb{Z}$  (greatest common divisor);
- primality:  $p$  is prime means  $p \in \mathbb{N}$ ,  $p \geq 2$ , and its only positive divisors are 1 and  $p$ .

## 2.2 Minimal discrete facts used (proved herein)

We explicitly prove the discrete facts we use:

- divisibility subtraction (Lemma [theorem 6.1](#));
- sign-invariance of  $\gcd$  (Lemma [theorem 6.2](#));
- transfer of divisibility through a multiple (Lemma [theorem 6.3](#));
- prime divisors exist: every  $d \geq 2$  has a prime divisor (Lemma [theorem 6.4](#));
- tail monotonicity of  $(a_i)$  (Lemma [theorem 4.1](#));
- a prime cannot divide a smaller nonzero integer (Lemma [theorem 4.2](#)).

The sole analytic input is Euler's theorem that  $\sum_{p \text{ prime}} 1/p$  diverges; Appendix [A](#) gives a detailed proof sketch and references [[Apo76](#), [HW08](#)].

## 3 Problem statement

Fix  $n \in \mathbb{N}$ . Define integers  $a_0 = n$ ,  $a_1 = 1$ , and for each  $k \geq 2$  define  $a_k$  to be the least integer  $m > a_{k-1}$  satisfying

$$\gcd(n - m, n - a_i) = 1 \quad \forall 1 \leq i < k.$$

Define

$$S(n) := \sum_{i \geq 1} \frac{1}{a_i}.$$

**Definition 3.1** (Good index / subseries). An index  $i \geq 1$  is **good** if there exists a prime  $p$  such that

$$p \leq a_i \quad \text{and} \quad p \mid (n - a_i).$$

Let  $S_{\leq}(n)$  be the subseries of  $S(n)$  over good indices, and let  $S_{>}(n)$  be the complementary subseries.

## 4 Difference-sequence reformulation and basic monotonicity

For  $i \geq 1$ , define

$$b_i := n - a_i \in \mathbb{Z}.$$

Then the greedy condition is equivalent to:

At step  $k \geq 2$ , choose the least  $a_k > a_{k-1}$  such that the new difference  $b_k = n - a_k$  satisfies  $\gcd(b_k, b_i) = 1$  for all  $1 \leq i < k$ .

**Lemma 4.1** (Tail monotonicity). *For all  $k \geq 2$ , one has  $a_k > a_{k-1}$ . In particular, the tail  $(a_i)_{i \geq 1}$  is strictly increasing:  $a_1 < a_2 < a_3 < \dots$ .*

*Proof.* This is immediate from the definition of  $a_k$  as the least integer strictly greater than  $a_{k-1}$  satisfying the gcd constraints.  $\square$

**Lemma 4.2** (A prime cannot divide a smaller nonzero integer). *If  $q$  is prime,  $x \in \mathbb{Z}$ ,  $x \neq 0$ , and  $|x| < q$ , then  $q \nmid x$  and hence  $\gcd(q, x) = 1$ .*

*Proof.* If  $q \mid x$ , then  $x = qt$  for some  $t \in \mathbb{Z}$ . Since  $x \neq 0$ , we have  $t \neq 0$  and thus  $|x| = |qt| \geq q$ , contradicting  $|x| < q$ . Therefore  $q \nmid x$ . Because  $q$  is prime, the only positive divisors of  $q$  are 1 and  $q$ , and since  $q \nmid x$ , the greatest common divisor must be 1.  $\square$

## 5 Edge cases: $n = 1$ and $n = 2$

A robust solution must explicitly isolate the terminating cases.

### 5.1 $n = 2$ (termination)

We have  $a_1 = 1$ . The least integer  $> 1$  such that  $\gcd(2 - a_2, 2 - 1) = 1$  is  $a_2 = 2$  because  $\gcd(0, 1) = 1$ . Then  $b_2 = 2 - a_2 = 0$ . Any further choice would require  $\gcd(2 - a_k, 0) = 1$ , i.e.  $|2 - a_k| = 1$ , forcing  $a_k \in \{1, 3\}$ . Only  $a_3 = 3$  is possible, and then no  $a_4 > 3$  remains admissible. Thus

$$S(2) = 1 + \frac{1}{2} + \frac{1}{3} < \infty.$$

### 5.2 $n = 1$ (termination)

We have  $a_1 = 1$  and hence  $b_1 = 1 - a_1 = 0$ . At step  $k = 2$ , admissibility requires  $\gcd(1 - a_2, 0) = 1$ , i.e.  $|1 - a_2| = 1$ , forcing  $a_2 \in \{0, 2\}$ . Since  $a_2 > a_1 = 1$ , we must have  $a_2 = 2$ , so  $b_2 = 1 - a_2 = -1$ . At step  $k = 3$ , admissibility must again include  $\gcd(1 - a_3, 0) = 1$ , forcing  $a_3 \in \{0, 2\}$ , but we also require  $a_3 > a_2 = 2$ , impossible. Thus the process terminates for  $n = 1$  as well, and  $S(1) < \infty$ .

## 6 Discrete gcd/divisibility lemmas used implicitly

For mechanical clarity we isolate standard facts used later.

**Lemma 6.1** (Divisibility subtraction). *If  $d, x, y \in \mathbb{Z}$  and  $d \mid x$  and  $d \mid y$ , then  $d \mid (x - y)$ .*

*Proof.* Write  $x = du$  and  $y = dv$  for some  $u, v \in \mathbb{Z}$ . Then  $x - y = d(u - v)$ , so  $d \mid (x - y)$ .  $\square$

**Lemma 6.2** (Sign invariance of gcd). *For all  $x, y \in \mathbb{Z}$ ,  $\gcd(-x, y) = \gcd(x, y) = \gcd(x, -y)$ .*

*Proof.* The common divisors of  $(x, y)$  and  $(-x, y)$  are the same because  $d \mid x$  iff  $d \mid (-x)$ . Therefore the greatest common divisor is unchanged by sign.  $\square$

**Lemma 6.3** (Divisor transfer through a multiple). *If  $d, b, B \in \mathbb{Z}$  with  $d \mid b$  and  $b \mid B$ , then  $d \mid B$ .*

*Proof.* Since  $b \mid B$ , write  $B = bt$ . Since  $d \mid b$ , write  $b = ds$ . Then  $B = (ds)t = d(st)$ , hence  $d \mid B$ .  $\square$

**Lemma 6.4** (Prime divisors exist). *If  $d \in \mathbb{N}$  and  $d \geq 2$ , then  $d$  has a prime divisor.*

*Proof.* Let  $S := \{m \in \mathbb{N} : m \mid d, m \geq 2\}$ . This set is nonempty (since  $d \in S$ ). Let  $p$  be the least element of  $S$  (well-ordering of  $\mathbb{N}$ ). If  $p$  were not prime, then  $p = uv$  with  $1 < u < p$  and  $1 < v < p$ . Then  $u \mid p \mid d$ , so  $u \in S$  with  $2 \leq u < p$ , contradicting minimality of  $p$ . Hence  $p$  is prime and divides  $d$ .  $\square$

## 7 Nonzero differences for $n \geq 3$

**Lemma 7.1** (No hit of  $a = n$  for  $n \geq 3$ ). *If  $n \geq 3$ , then  $a_k \neq n$  for all  $k \geq 2$  (equivalently,  $b_k \neq 0$  for all  $k \geq 2$ ).*

*Proof.* Since  $a_1 = 1$ , we have  $b_1 = n - 1 \geq 2$ . If some  $a_k = n$ , then  $b_k = 0$  and admissibility would require

$$1 = \gcd(b_k, b_1) = \gcd(0, n - 1) = n - 1,$$

impossible for  $n \geq 3$ . □

## 8 Well-definedness of the greedy recursion for $n \geq 3$

The definition of  $a_k$  as a *least admissible* integer requires proving admissible integers exist at each stage and that choosing the least one preserves the invariants.

**Definition 8.1** (Admissible at stage  $k$ ). Fix  $n \in \mathbb{N}$ . Suppose  $a_1, \dots, a_{k-1}$  are defined with  $k \geq 2$ . An integer  $m \in \mathbb{N}$  is **admissible at stage  $k$**  if  $m > a_{k-1}$  and

$$\gcd(n - m, n - a_i) = 1 \quad \forall 1 \leq i < k.$$

**Lemma 8.2** (Pairwise coprimality invariant). *Assume  $a_1, \dots, a_k$  have been chosen so that for each  $2 \leq j \leq k$  we have  $\gcd(n - a_j, n - a_i) = 1$  for all  $1 \leq i < j$ . Then the differences  $b_i := n - a_i$  are pairwise coprime: for all distinct  $i, j$  with  $1 \leq i, j \leq k$ ,*

$$\gcd(b_i, b_j) = 1.$$

*Proof.* If  $i < j$  then the defining condition at step  $j$  gives  $\gcd(b_j, b_i) = 1$ . Since gcd is symmetric,  $\gcd(b_i, b_j) = 1$  as well. □

**Lemma 8.3** (Euclid-type witness guarantees a new admissible element). *Assume  $n \geq 3$  and we have chosen  $a_1 < \dots < a_{k-1}$  such that  $b_i = n - a_i \neq 0$  and  $\gcd(b_i, b_j) = 1$  for  $i \neq j$ . Let*

$$B := \prod_{i=1}^{k-1} |b_i| \in \mathbb{N}, \quad m := n + (B + 1).$$

*Then  $m > a_{k-1}$  and  $\gcd(n - m, b_i) = 1$  for all  $1 \leq i < k$ .*

*Proof.* We have  $n - m = -(B + 1)$ . By Lemma [theorem 6.2](#),  $\gcd(n - m, b_i) = \gcd(B + 1, b_i)$ . Suppose a number  $d > 1$  divides both  $B + 1$  and  $b_i$ . Since  $b_i \mid B$ , Lemma [theorem 6.3](#) gives  $d \mid B$ . Then Lemma [theorem 6.1](#) implies  $d \mid (B + 1) - B = 1$ , impossible. Hence  $\gcd(B + 1, b_i) = 1$ , and thus  $\gcd(n - m, b_i) = 1$ .

Also  $B \geq |b_{k-1}| = |n - a_{k-1}|$ . If  $a_{k-1} \geq n + 1$ , then  $|n - a_{k-1}| = a_{k-1} - n$  and  $m = n + (B + 1) \geq n + (a_{k-1} - n + 1) = a_{k-1} + 1 > a_{k-1}$ . If  $a_{k-1} \leq n - 1$ , then  $m \geq n + 2 > a_{k-1}$ . □

**Theorem 8.4** (Well-defined infinite greedy recursion for  $n \geq 3$ ). *Fix  $n \geq 3$ . The greedy rule defines an infinite sequence  $(a_k)_{k \geq 0}$  with  $a_0 = n$ ,  $a_1 = 1$  such that the following properties hold.*

1. For every  $k \geq 2$ ,  $a_k$  is the least integer  $> a_{k-1}$  satisfying  $\gcd(n - a_k, n - a_i) = 1$  for all  $1 \leq i < k$ .

2. The tail  $(a_i)_{i \geq 1}$  is strictly increasing.
3. The differences  $b_i := n - a_i$  for  $i \geq 1$  are pairwise coprime, and for  $i \geq 2$  they are nonzero.

*Proof.* We proceed inductively on  $k \geq 1$  constructing  $a_{k+1}$  from  $a_1, \dots, a_k$ .

Induction hypothesis:  $a_1, \dots, a_k$  are defined, strictly increasing, and satisfy the admissibility condition for each step up to  $k$ . By Lemma [theorem 8.2](#), the differences  $b_i = n - a_i$  are pairwise coprime for  $1 \leq i \leq k$ . By Lemma [theorem 7.1](#), for  $n \geq 3$  we have  $a_j \neq n$  for all  $j \geq 2$ , hence  $b_j \neq 0$  for  $2 \leq j \leq k$ .

By Lemma [theorem 8.3](#), there exists an integer  $m > a_k$  with  $\gcd(n - m, b_i) = 1$  for all  $1 \leq i \leq k$ ; i.e. an admissible integer at stage  $k + 1$  (Definition [theorem 8.1](#)). Therefore the set of admissible integers at stage  $k + 1$  is nonempty, and by the well-ordering of  $\mathbb{N}$  it has a least element. Define  $a_{k+1}$  to be this least admissible integer.

By construction,  $a_{k+1} > a_k$ , so strict increase continues. Also  $a_{k+1}$  satisfies the gcd constraints against all earlier indices, so the defining property holds at step  $k + 1$ . This closes the induction and yields an infinite sequence.  $\square$

**Corollary 8.5** (Unboundedness). *For every  $n \geq 3$  and every  $M \in \mathbb{N}$  there exists  $i \geq 1$  with  $a_i \geq M$ .*

*Proof.* From Theorem [theorem 8.4](#) we have  $a_{i+1} \geq a_i + 1$  for all  $i \geq 1$ , and  $a_1 = 1$ . By induction  $a_i \geq i$  for all  $i \geq 1$ . Taking  $i = M$  gives  $a_i \geq M$ .  $\square$

## 9 Prime embedding theorem (the core engine)

The proof uses a clean “first-crossing” lemma to make the minimality step explicit.

**Lemma 9.1** (First-crossing lemma). *Fix  $n \in \mathbb{N}$  and let  $(a_i)$  be a greedy sequence. Let  $M \in \mathbb{N}$  and let  $k \geq 2$  be the least index such that  $a_k \geq M$ . If  $M$  is admissible at stage  $k$  (i.e.  $M > a_{k-1}$  and  $\gcd(n - M, n - a_i) = 1$  for all  $1 \leq i < k$ ), then  $a_k = M$ .*

*Proof.* Since  $k$  is least with  $a_k \geq M$ , we have  $a_{k-1} < M$ . By assumption,  $M$  is admissible at stage  $k$ . But  $a_k$  is, by definition, the least admissible integer  $> a_{k-1}$ . Therefore  $a_k \leq M$ . Together with  $a_k \geq M$ , we get  $a_k = M$ .  $\square$

**Theorem 9.2** (Prime embedding). *Let  $n \geq 3$ . For every prime  $q$  with  $q > n - 1$ , the value  $a = n + q$  occurs in the greedy sequence; i.e.  $\exists i \geq 1$  with  $a_i = n + q$ .*

*Proof.* Fix a prime  $q > n - 1$  and set  $M := n + q$ . By Corollary [theorem 8.5](#) there exists an index  $k \geq 2$  with  $a_k \geq M$ ; let  $k$  be the least such index. Then  $a_{k-1} < M$ , so  $M > a_{k-1}$ .

We claim  $M$  is admissible at stage  $k$ . Let  $i$  satisfy  $1 \leq i < k$ . By Lemma [theorem 4.1](#),  $a_i \leq a_{k-1} < M = n + q$ , so  $a_i < n + q$ . Define  $b_i := n - a_i$  and note  $n - M = -q$ .

We show  $\gcd(n - M, b_i) = 1$ . Split into cases:

- If  $a_i \leq n$ , then  $b_i = n - a_i$  is a positive integer with  $1 \leq b_i \leq n - 1 < q$ .
- If  $a_i > n$ , then  $b_i < 0$  and  $|b_i| = a_i - n < q$  because  $a_i < n + q$ .

If  $i \geq 2$ , Lemma [theorem 7.1](#) gives  $b_i \neq 0$ . If  $i = 1$ , then  $b_1 = n - 1 \geq 2$ , so also  $b_1 \neq 0$ . Thus  $b_i \neq 0$  for all  $1 \leq i < k$ . By Lemma [theorem 4.2](#),  $q \nmid b_i$  and hence  $\gcd(q, b_i) = 1$ . By Lemma [theorem 6.2](#),  $\gcd(-q, b_i) = \gcd(q, b_i) = 1$ , i.e.  $\gcd(n - M, n - a_i) = 1$ .

So  $M$  is admissible at stage  $k$ . By Lemma [theorem 9.1](#),  $a_k = M = n + q$ .  $\square$

## 10 Divergence of $S(n)$ for all $n \geq 3$

**Theorem 10.1** (Main divergence). *For every  $n \geq 3$ , the series  $S(n) = \sum_{i \geq 1} 1/a_i$  diverges to  $+\infty$ .*

*Proof.* By Theorem [theorem 9.2](#), for each prime  $q > n$  some term equals  $n + q$ . Hence

$$S(n) = \sum_{i \geq 1} \frac{1}{a_i} \geq \sum_{\substack{q \text{ prime} \\ q > n}} \frac{1}{n + q}.$$

Now for  $q > n$  we have  $n + q < 2q$ , so  $\frac{1}{n+q} > \frac{1}{2q}$ . Therefore

$$S(n) \geq \frac{1}{2} \sum_{\substack{q \text{ prime} \\ q > n}} \frac{1}{q}.$$

Euler's theorem ([Theorem theorem A.4](#)) states that  $\sum_{q \text{ prime}} 1/q$  diverges; removing finitely many terms from a divergent series of nonnegative terms leaves a divergent tail, so  $\sum_{q > n} 1/q$  diverges as well. Hence  $S(n) = +\infty$ .  $\square$

*Remark 10.2* (Boundary case  $q = n$ ). If  $n$  itself is prime, then the set  $\{q \text{ prime} : q > n - 1\}$  includes  $q = n$ . The single term  $1/(n + n) = 1/(2n)$  does not affect divergence. We avoid this boundary by summing over  $q > n$  in [Theorem theorem 10.1](#).

*Remark 10.3* (As  $n \rightarrow \infty$ ). Since  $S(n) = +\infty$  for every  $n \geq 3$ , the function  $n \mapsto S(n)$  is identically  $+\infty$  on  $\{3, 4, 5, \dots\}$ . In particular,  $S(n) \rightarrow \infty$  as  $n \rightarrow \infty$  (in the extended real sense).

## 11 Restricted sums: $S_{\leq}(n)$ diverges and $S_{>}(n)$ is finite

### 11.1 The “good” subseries diverges for $n \geq 3$

**Theorem 11.1** (Good subseries divergence). *For every  $n \geq 3$ , the good subseries  $S_{\leq}(n)$  diverges to  $+\infty$ .*

*Proof.* Take any prime  $q > n$ . By [Theorem theorem 9.2](#), some term equals  $a = n + q$ . Then  $n - a = -q$ , which is divisible by the prime  $q$ , and clearly  $q \leq n + q = a$ . Thus that index is good ([Definition theorem 3.1](#)), and

$$S_{\leq}(n) \geq \sum_{\substack{q \text{ prime} \\ q > n}} \frac{1}{n + q} = +\infty,$$

since the right-hand side diverges by the same comparison used in [Theorem theorem 10.1](#).  $\square$

### 11.2 The “bad” subseries is always finite (all $n$ )

**Lemma 11.2** (All sufficiently large indices are good). *If  $i \geq 1$  and  $a_i > n + 1$ , then  $i$  is good.*

*Proof.* Let  $d := a_i - n \geq 2$ . Then  $n - a_i = -d$ . By [Lemma theorem 6.4](#),  $d$  has a prime divisor  $p$ . Then  $p \mid d$  implies  $p \mid (n - a_i)$ . Also  $p \leq d < a_i$ , hence  $p \leq a_i$ . Therefore  $i$  is good ([Definition theorem 3.1](#)).  $\square$

**Corollary 11.3** (Finiteness of the bad part). *For every  $n \in \mathbb{N}$ , the subseries  $S_{>}(n)$  is finite (hence convergent).*

*Proof.* By Lemma [theorem 11.2](#), if  $i$  is bad then  $a_i \leq n + 1$ . By Lemma [theorem 4.1](#),  $(a_i)_{i \geq 1}$  is strictly increasing, so there are only finitely many  $i \geq 1$  with  $a_i \leq n + 1$ . Thus  $S_{>}(n)$  has only finitely many nonzero terms, hence is finite.  $\square$

## 12 Final theorem

**Theorem 12.1** (Complete solution to Erdős #460). *The following statements hold.*

1. For  $n = 1, 2$ , the greedy sequence terminates and  $S(n) < \infty$ .
2. For every  $n \geq 3$ , the series  $S(n) = \sum_{i \geq 1} 1/a_i$  diverges to  $+\infty$ .
3. For every  $n \geq 3$ , the good subseries  $S_{\leq}(n)$  diverges to  $+\infty$ .
4. For every  $n \in \mathbb{N}$ , the bad subseries  $S_{>}(n)$  is finite (hence convergent).
5. Consequently,  $S(n) \rightarrow \infty$  as  $n \rightarrow \infty$  (see Remark [theorem 10.3](#)).

## A Appendix A: divergence of $\sum_p \text{prime } 1/p$

This appendix gives a detailed proof sketch (Euler product / harmonic comparison). For complete treatments see [[Apo76](#), [HW08](#)].

**Proposition A.1** (Harmonic divergence). *The harmonic series  $H_N = \sum_{m=1}^N \frac{1}{m}$  satisfies  $H_N \rightarrow \infty$  as  $N \rightarrow \infty$ .*

*Proof.* Group terms by powers of 2:

$$H_{2^t} = 1 + \sum_{j=1}^t \sum_{m=2^{j-1}+1}^{2^j} \frac{1}{m} \geq 1 + \sum_{j=1}^t 2^{j-1} \cdot \frac{1}{2^j} = 1 + \frac{t}{2} \rightarrow \infty.$$

$\square$

*Remark A.2* (A standard log inequality). For  $0 < u \leq \frac{1}{2}$ , one has  $-\ln(1-u) \leq 2u$ . One proof: use the integral representation  $-\ln(1-u) = \int_0^u \frac{1}{1-t} dt \leq \int_0^u \frac{1}{1-u} dt = \frac{u}{1-u} \leq 2u$ .

**Proposition A.3** (Euler comparison). *For  $N \geq 2$ , define*

$$P(N) := \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1}.$$

*Then  $P(N) \geq H_N := \sum_{m=1}^N \frac{1}{m}$ .*

*Proof sketch.* Expand each factor as a geometric series:

$$\left(1 - \frac{1}{p}\right)^{-1} = 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots.$$

Multiplying finitely many such series gives a sum of terms of the form  $1/m$ , where  $m$  ranges over positive integers whose prime factors are all  $\leq N$ . In particular, every integer  $1 \leq m \leq N$

has all prime factors  $\leq m \leq N$ , so the term  $1/m$  appears in this expansion with nonnegative coefficient. Therefore  $P(N) \geq \sum_{m=1}^N 1/m = H_N$ .  $\square$

**Theorem A.4** (Euler:  $\sum_{p \text{ prime}} 1/p$  diverges). *The series  $\sum_{p \text{ prime}} \frac{1}{p}$  diverges.*

*Proof sketch.* By Proposition [theorem A.3](#),  $P(N) \geq H_N$ . Taking logs,

$$\ln P(N) = \sum_{p \leq N} -\ln\left(1 - \frac{1}{p}\right).$$

For  $p \geq 2$  we have  $0 < \frac{1}{p} \leq \frac{1}{2}$ , so by Remark [theorem A.2](#),  $-\ln(1 - \frac{1}{p}) \leq \frac{2}{p}$ . Hence

$$\ln P(N) \leq 2 \sum_{p \leq N} \frac{1}{p}.$$

Combining with  $\ln P(N) \geq \ln H_N$  gives

$$2 \sum_{p \leq N} \frac{1}{p} \geq \ln H_N.$$

By Proposition [theorem A.1](#),  $H_N \rightarrow \infty$ , hence  $\ln H_N \rightarrow \infty$ , forcing  $\sum_{p \leq N} \frac{1}{p} \rightarrow \infty$ .  $\square$

## References

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